Longitudinal Elevation Profiles of Rivers: Curve Fitting with Functions Predicted by Theory

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***Abstract*—This paper provides a concise review of three different derivations for the shapes of longitudinal elevation profiles in rivers. These are then briefly compared to an observed elevation profile for Beaver Creek, Kentucky, as extracted from a 1-arcsecond DEM.**

I. INTRODUCTION

Longitudinal elevation profiles in rivers are generally concave down for a short distance from the drainage divide and are then concave up over the rest of the profile. This has generally been interpreted as due to a change in process dominance between essentially diffusive processes near divides and fluvial processes that dominate once discharge values become large enough. There have been many efforts to explain the shapes of these profiles from physical and empirical laws. This paper provides a concise review of three different derivations for the shapes of longitudinal profiles and then briefly compares them to observed profiles for Beaver Creek, Kentucky.

II. LONGITUDINAL PROFILES FROM AN IDEALIZED STEADY-STATE FLUVIAL LANDFORM MODEL

To explore connections between function and form in fluvial landscapes, Peckham (1999, 2003a, 2003b, 2003c) studied a partial differential equation intended as a simplified model for steady-state fluvial landforms. This equation is derived by combining three assumptions for the *steady-state* flow of water over a landscape: (1) mass conservation for water, (2) 2D flow direction given by the antigradient of the water's free surface and (3) a power-law relationship between unit-width discharge, q, and free surface slope, S. These can be expressed mathematically as: (1) $\nabla \cdot \underline{q} = R$, (2) $\underline{q} = -q\,(\nabla z / S)$ and (3) $|\underline{q}| = q = q_1 S^{\gamma}$. The third assumption comes from combining two well-known empirical equations of downstream hydraulic geometry (Leopold et al., 1995): $S \propto Q^{\alpha}$ and $w \propto Q^{b}$, where $Q = q\,w$ is volumetric discharge, q is unit-width discharge, $S = S(x,y) = |\nabla z| = (z_x^2 + z_y^2)^{1/2} \geq 0$ is the slope at the point (x,y), and w is channel width. This implies that $\gamma = (1-b)/\alpha < 0$, and $\gamma \approx -1$ for typical α and b values. Together, these equations lead to a nonlinear, 2nd-order PDE for the free water surface, $z(x,y)$:

$$\nabla \cdot \left(S^{\gamma-1} \nabla z\right) = -R^{\star} \tag{1}$$

where, $R^{\star} = R/q_1$. Here R is a *geomorphically effective rainrate* (assumed steady and spatially uniform) and q_1 is the unit-width discharge that corresponds to unit slope ($S = 1$). R is viewed as an extreme and geologically rare rainfall rate that produces flows with extreme shear stresses that can reshape the landscape. It is assumed to last long enough that shear stresses on the reshaped landscape drop to subcritical levels everywhere as a result of readjustment (e.g. wider channels, higher sinuosities, redistribution of roughness elements and formation of bedforms). Erosion caused by lesser rainfall rates is assumed to be insignificant by comparison. Sediment transport is not represented directly; the idea is to seek a landscape that has adjusted so that runoff rates less than or equal to R are insufficient to exceed the critical shear stress required to initiate significant sediment transport. While Peckham (1999, 2003) obtained numerous results for the 2D version of this equation, the 1D version (taking $z = z(x)$) is given by:

$$\left(S^{\gamma-1} z'(x)\right)' = -R^{\star} \tag{2}$$

Since $z(x)$ is a decreasing function and $S(x) \geq 0$, we have $S(x) = -z'(x)$, so (2) simplifies to $(S^{\gamma})' = R^{\star}$. Integrating with respect to x and rearranging terms we have

$$z'(x) = -\left[S_0^{\gamma} + R^{\star}\,(x - x_0)\right]^{\frac{1}{\gamma}}, \tag{3}$$

In: Geomorphometry for Geosciences, Jasiewicz J., Zwoliński Zb., Mitasova H., Hengl T. (eds), 2015. Adam Mickiewicz University in Poznań - Institute of Geoecology and Geoinformation, International Society for Geomorphometry, Poznań

where $S_0 = -z'(x_0)$. In the case where $\gamma \neq -1$, a second integration yields the general solution:

$$z(x) = z_0 + \frac{1}{p_\gamma R^\star} \left\{ S_0^{\gamma+1} - [S_0^\gamma + R^\star (x - x_0)]^{p_\gamma} \right\} \quad (4)$$

where $p_\gamma = (\gamma + 1)/\gamma$. Note that $p_\gamma \leq 0$ for $\gamma \in [-1, 0)$ and is positive otherwise. In the case where $\gamma = -1$, the second integration of (3) gives:

$$z(x) = z_0 - \frac{1}{R^\star} \left\{ \ln \left[1 + S_0 \, R^\star (x - x_0) \right] \right\}. \quad (5)$$

In (4) and (5) we have $z(x_0) = z_0$ and $z'(x_0) = -S_0$.

III. Steady-state Longitudinal Profiles from a Sediment Transport Law and Hack's Law

A generalized and widely-used sediment transport law – which contains several other transport laws for specific choices of m and n – is given by

$$Q_s = K Q^m S^n. \quad (6)$$

The coefficient, K, is usually called the *erodibility* and the exponents m and n are both typically taken to be between 1 and 2. Assuming that the long-term average runoff rate, R is spatially uniform, we have $Q = RA$, where A is the total contributing area above a given river outlet. Assuming a steady, spatially uniform rate of tectonic uplift, U, that exactly balances the rate of fluvial erosion (sometimes called "dynamic equilibrium"), we must similarly have $Q_s = UA$. Combining these with (6), we therefore find

$$S_{eq} = [U/(KR^m)]^{1/n} \, A^{(1-m)/n} \quad (7)$$

where S_{eq} is the steady-state or equilibrium slope at any point and R is a uniform runoff rate (Tucker and Bras, 1998). This agrees with an empirical law called Flint's Law (Flint, 1974), often written as $S = cA^{-\theta}$, where the exponent $\theta > 0$ is typically close to $1/2$. The prediction here is that $\theta = (m-1)/n$, and $\theta > 0$ since $m > 1$. As shown by Whipple and Tucker (1999), an empirical law known as Hack's Law, given by

$$A = k_h \, x^h \quad (8)$$

can be inserted into (7) to express slope in terms of x, the flow distance downstream from a drainage divide. The Hack exponent is typically close to $3/5 = 0.6$, so its reciprocal, h, is close to $5/3 = 1.67$. This results in the equation: $S(x) = -z'(x) = Cx^{p-1}$, which can be integrated to get a functional form for the longitudinal profile. This gives

$$z(x) = \begin{cases} z_0 - (C/p)\,(x^p - x_0^p), & \text{if } p \neq 0 \\ z_0 - C \ln (x/x_0) & \text{if } p = 0, \end{cases} \quad (9)$$

where $p = 1 - \theta h$ and $C = [U/(KR^m)]^{1/n} k_h^{-\theta}$. For $\theta = 1/2$ and $h = 5/3$ we have $p = 1/6$. Note that $z(x_0) = z_0$ and $z'(x_0) = -S_0$, where $S_0 = Cx_0^{p-1}$. These functional forms are somewhat similar to those predicted from the idealized steady-state fluvial landform model, as given by (4) and (5). However, here $p > 0$ and there, $p_\gamma < 0$. Also, here we need $x_0 \neq 0$ to avoid an infinite slope at x_0. Despite these differences, their plots look very similar for typical values of their parameters.

IV. Longitudinal Profiles from a Sediment Transport Law and Similarity Solutions

Smith et al. (2000) developed a theory of graded streams that is based on conservation of sediment and the generalized sediment transport law

$$q_s = C x^\gamma S^\delta \quad (10)$$

where q_s is unit-width sediment discharge, x is downstream distance from a divide, C is a constant and $S = -z_x > 0$. While Smith et al. (2000) gave results for both (transport-limited) alluvial channel profiles and (detachment-limited) bedrock channel profiles, which obey different sediment conservation laws, here we restrict attention to alluvial channel profiles. For alluvial channels, conservation of sediment mass can be expressed as

$$z_t = \nabla \cdot [-q_s \, (\nabla z / S)] \quad (11)$$

which in the 1D case, after inserting (10) becomes

$$z_t = -C \left[x^\gamma \, (-z_x)^\delta \right]_x. \quad (12)$$

Similarity solutions to (12) are sought by inserting

$$z(x,t) = \tau^\alpha \, F \left(x / \tau^\beta \right) \quad (13)$$

and solving the resulting ODE for $F(\eta)$, where $\eta = x/\tau^{\beta}$ is called the *similarity variable* and $\tau = \tau_0 + \omega t$. The resulting solutions have an initial elevation of $z_0 = z(0,0)$ at the upstream end where $x = 0$ and descend to $z = 0$ at the downstream end where $x = X(t)$. An expression for $X(t)$ can be obtained by solving $z(X,t) = 0$ for X. The initial x-position is denoted by $X_0 = X(0)$.

Not including a tectonic uplift term in (12) allows similarity solutions to be found. However, if $z(x,t)$ is a similarity solution to (12), then it is easy to check that $z(x,t) + \int_0^t U(t)dt$ is a solution to (12) with $U = U(t)$ added to its right-hand side, where $U(t)$ is a spatially uniform uplift rate. The resulting solution will no longer be a similarity solution. For steady uplift, $U(t) = u_0$, and the solution is $z(x,t) + u_0 t$.

Smith et al. (2000) showed how these similarity solutions for alluvial channel profiles can be divided into four classes, each with a different physical interpretation. In all cases, *admissible* solutions require $\delta > 1$ and $0 < \gamma < \delta+1$. These are briefly summarized in the following four sections.

1) Fans and Pediments: $\alpha + \beta = 0$.
These solutions are characterized by zero net sediment loss, so that the mass of material under the profile does not change over time. They are always concave upwards and there is no net loss of material through the lower boundary, which advances downstream over time. Solutions are interpreted as representing a spreading fan of material, as occurs in internal basins such as those in arid and semi-arid regions of the western U.S..

$$z(x,t) = z_0\, G(\omega t)\, \{1 - [(x/X_0)\, G(\omega t)]^{p_1}\}^{p_2} \tag{14}$$

$$G(\omega t) = (1 + \omega t)^{1/(\gamma - 2\delta)} \tag{15}$$

$$\omega = -C\,(\gamma - 2\delta)\left(\frac{1-\gamma+\delta}{\delta - 1}\right)^{\delta} \frac{z_0^{\delta-1}}{X_0^{1-\gamma+\delta}} \tag{16}$$

where $p_1 = (1-\gamma+\delta)/\delta$, and $p_2 = \delta/(\delta-1)$. Admissible solutions have $(0 < p_1 < 1)$ and $p_2 > 1$.

2) Hanging Valleys: $\alpha = 0$.
These solutions are characterized by unconstrained sediment removal. The elevation at their upstream end is fixed for all time; that is, $z(0,t) = z_0$ for all t. Profiles are concave upwards for $\gamma > 2$, linear for $\gamma = 2$ and convex upwards for $\gamma < 2$. Slopes decrease over time if the profile is concave upwards and increase over time if it is convex or linear. $X(t)$ is a decreasing function, so the lower boundary moves upstream over time. Sediment discharge at the lower boundary remains constant over time. After a time, $t = \tau_0$, profiles approach a vertical "cliff" at $x = 0$.

$$z(x,t) = z_0\,[1 - G(\omega t)\,(x/X_0)^{p}] \tag{17}$$

$$G(\omega t) = (1 + \omega t)^{1/(1-\delta)} \tag{18}$$

$$\omega = -C\,(2\delta - \gamma)\left(\frac{1+\delta-\gamma}{\delta - 1}\right) \frac{z_0^{\delta-1}}{X_0^{1+\delta-\gamma}} \tag{19}$$

where $p = (1+\delta-\gamma)/(\delta-1) > 0$.

3) Fixed Lower Boundaries and Base Levels: $\beta = 0$.
These solutions are applicable to rivers that drain to a large water body with a fixed elevation. The x-position of their downstream end is fixed for all time; that is, $X = X_0$ and $z(X_0, t) = 0$ for all t. The solutions are *separable*, so that $z(x,t) = F(x)T(t)$, where

$$T(t) = [(\delta - 1)C\lambda\, t + 1]^{\frac{1}{1-\delta}} \tag{20}$$

$$\left[x^{\gamma}\,(-F_x)^{\delta}\right]_x = \lambda F. \tag{21}$$

Note that $\lambda > 0$, $T(0) = 1$ and $z(x,0) = F(x)$. However, the ODE for $F(x)$ must be solved numerically. Hypsometric curves for these solutions are time-invariant.

4) Steady-State Profiles and Tectonic Motion: $\alpha = 1$.
These solutions are characterized by $z_t = -L$, where L is a spatially uniform rate of downcutting. That is, solutions are curves that rise or lower at a constant rate and they are always upward concave, with

$$z(x,t) = z_0\,[(1 + \omega t) - (x/X_0)^{p}] \tag{22}$$

$$\omega = (-C/z_0)(z_0\, p\, X_0^{-p})^{\delta} < 0 \tag{23}$$

where $p = 1/\beta = (1+\delta-\gamma)/\delta$ and $0 < p < 1$. When the effect of steady, uniform uplift is included by adding $u_0 t$ (as explained previously), the terms $u_0 t$ and $z_0\, \omega t$ can be grouped, $z_t = -L = u_0 + z_0\, \omega$ and the result is still a similarity solution. In the special case where $u_0 = -\omega z_0$, we have $L = 0$ and the profile is a *steady-state* solution. This is a type of *dynamic equilibrium*, as discussed earlier.

V. Fitting Curves to Longitudinal Profiles

RiverTools 4.1 is a software toolkit for terrain analysis. It includes tools for extracting longitudinal elevation profiles from DEMs and finding best-fit parameters for a variety of functional forms using nonlinear least-squares regression. *Downstream profiles* from any cell in the DEM can be extracted, in addition to *upstream* or *main channel profiles*. The latter are defined by repeatedly moving upstream toward the D8 neighbor cell with the largest total contributing area until reaching a drainage divide. Figure 1 shows the best fit of equation (4) to the main channel profile for Beaver Creek, Kentucky. This function provides the smallest standard error of any of those tested ($\epsilon = 4.55$), with $\gamma = -0.70$ and $R^{\star} = 0.0035$. Note that $x_0 = 0$, $z_0 = 668.33$ and $S_0 = 0.462$ were held fixed. If we take $q_1 = 0.007$ $[m^2/s]$ (Leopold et al., 1995), this implies $R = 87.2$ $[mmph]$. This is a very large, but not unrealistic value. A rate of 435 $[mmph]$ was sustained for 42 minutes in Holt, Missouri (Lott, 1954). As explored by Peckham (2003c), this suggests a method for deducing the magnitude of landscape-shaping rainfall events from elevation data.

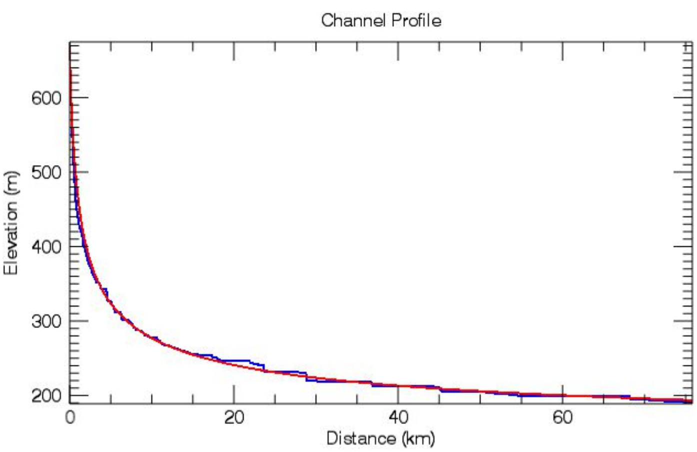

Fig. 1. Best fit of (4) to the main channel profile of Beaver Creek, KY.

Figure 2 shows the best fit of the power-law (9) to the main channel profile for Beaver Creek, Kentucky. This curve has $p = 0.133$, $C = 14.71$ (x_0 and z_0 again fixed). The standard error, $\epsilon = 15.93$, is 3.5 times larger. The best fit of the power-law (22) is identical, since $x_0 = 0$, with the same p and ϵ (steady case). An exponential curve constrained to go through (x_0, z_0) provides an extremely poor fit (not shown), with $\epsilon = 123.33$. An unconstrained, 3-parameter exponential curve, also poor, has $\epsilon = 15.49$.

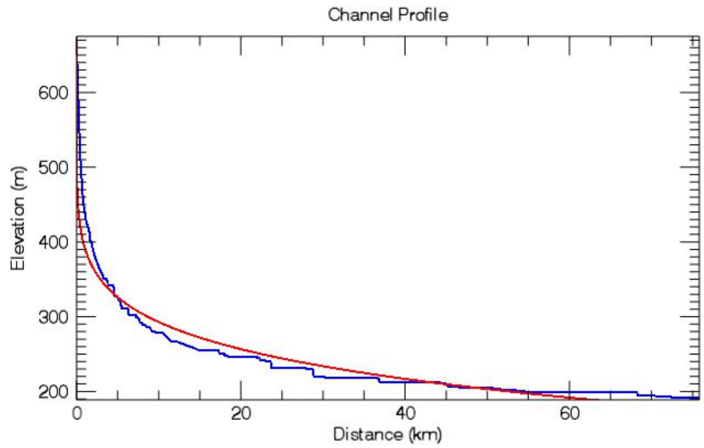

Fig. 2. Best fit of power-laws (9) and (22) to the main channel profile of Beaver Creek, KY. Standard error $= 15.93$ and $p = 0.133$.

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